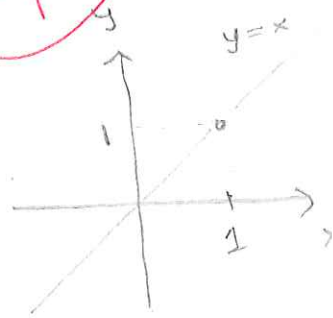
M413 - Fall 2013 - HW 9 - Enrique Areyan (1)

(1) Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies:

$$\lim_{h\to 0} [f(x+h) - f(x-h)] = 0. \quad \forall x \in \mathbb{R}.$$

60/60

Does this imply that f is continuous?



Solution: NO. the following is a counter example:

Define $f: \mathbb{R} \to \mathbb{R}$ to be $f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

claims: (a) $\lim_{h\to 0} [f(x+h) - f(x-h)] = 0 \quad \forall x \in \mathbb{R}$.

(b) f is not continuous at $x = 1$.

Pf: (a) By linearity of the limit operator on $\mathbb{R}$:

$$\lim_{h\to 0} [f(x+h) - f(x-h)] = \lim_{h\to 0} f(x+h) - \lim_{h\to 0} f(x-h)$$

$$= \lim_{h\to 0} (x+h) - \lim_{h\to 0} x-h \quad \text{By def of } f$$

$$= x - x = \boxed{0}$$

Formally, we can show that, for every $x \in \mathbb{R}$: $(x+h \neq 1)$

let $\varepsilon > 0$. Pick $\delta > 0$ to be such that $\delta < \varepsilon$. then

$|f(x+h) - x| = |x+h-x| = |h| < \delta < \varepsilon \Rightarrow |f(x+h) - x| < \varepsilon$

A similar argument works for the case $x+h = 1$

So the left and right limits of this function agree everywhere.

(b) $\lim_{x\to 1} f(x) = 1$ but $f(1) = 0$. By theorem 4.6, we have that f is not continuous at 1.

(a) & (b) show that f is a real function on $\mathbb{R}$ s.t

$$\lim_{h\to 0} [f(x+h) - f(x-h)] = 0 \quad \forall x \in \mathbb{R}, \text{ i.e., that left}$$

and right limits agree but, it is not continuous because it has a discontinuity at $x = 1$.

2) If f is a continuous mapping of a metric space X into a metric space Y, Prove that $f(\overline{E}) \subset \overline{f(E)}$, for every set $E \subset X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$

Pf: Let $f:(X,d_X) \to (Y,d_Y)$ be a continuous map. Let $E \subset X$. Suppose that $a \in f(\overline{E})$. By definition: $f(\overline{E}) = \{y \in Y \mid f(e)=y, \text{ some } e \in \overline{E}\}$. Therefore, there exists $e \in \overline{E}$ such that $f(e)=a$. Since by definition $\overline{E} = E \cup E'$, we need to consider two cases:

(i) $e \in E$. In this case $a \in f(E)$ and hence $a \in f(E) \cup f(E)'$ which is the same as $a \in \overline{f(E)}$, and we are done.

(ii) $e \in E'$. In this case e is a limit point of $f(E)$. therefore, for any $r>0$, $N_r(e)\setminus\{e\} \cap E \neq \Phi$. Let $x \in N_r(e)\setminus\{e\} \cap E$. Now use the fact that f is continuous, i.e., $\forall \varepsilon > 0 \; \exists \delta > 0$ such that $\forall q \in X$: If $d_X(q,p) < \delta$ then $d_Y(f(q),f(p)) < \varepsilon$. Note that $x \in N_r(e)\setminus\{e\}$, so $x \neq e$ and $d_X(x,e) < r$. choose $r=\delta$ to get that $d_Y(f(x),f(e)) < \varepsilon$. We are allowed to make this choice because r is arbitrary, so we let $\varepsilon > 0$, find δ and consider $x \in N_\delta(e)\setminus\{e\}$. But then $f(x) \in N_\varepsilon(f(e))$. But the choice of ε was arbitrary so $f(e)$ is a limit point of $f(E)$. (Note $f(x) \in N_\varepsilon(f(e))$ and $f(x) \in f(E)$ since $x \in E$; so $N_\varepsilon(f(e))\setminus\{f(e)\} \cap f(E) \neq \Phi$ for any $\varepsilon > 0$).

Hence, $f(e) = a \in f(E)' \Leftarrow a \in f(E) \cup f(E)' \Leftarrow a \in \overline{f(E)}$.

The following example shows that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$. Consider:

$$f(x) = \frac{1}{e^x}. \quad \text{take } E = (0,\infty) \Rightarrow \overline{E} = [0,\infty)$$

then:

$f(E) = (0,1) \Rightarrow \overline{f(E)} = [0,1]$ But

$f(\overline{E}) = (0,1]$. therefore $1 \in \overline{f(E)}$ but $1 \notin f(\overline{E})$

so the example shows that the inclusion can be proper.

M413 - Fall 2013 - HW9 - Enrique Areyan

(3) Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ s.t. $f(p)=0$. Prove that $Z(f)$ is closed.

Pf: Two different proofs:

(I) Since f is continuous, given $\varepsilon>0$ there exists $\delta>0$ s.t. $\forall x \in X$ If $d_X(x,p)<\delta$ then $d_X(f(x),f(p))<\varepsilon$.

Let p be a limit point of $Z(f)$. Let $\varepsilon>0$. Choose $\delta>0$ such that the continuity condition holds. Since p is a limit point, there exists $x \in N_\delta(p)\setminus\{p\} \cap Z(f)$. Hence, $d_X(x,p)<\delta$ (1st) and $x \in Z(f)$ (2nd).

the first implies $d_X(f(x),f(p))<\varepsilon \iff f(x) \in N_\varepsilon(f(p))$

the second implies $f(x)=0$

therefore, $0 \in N_\varepsilon(f(p)) \iff |0-f(p)|<\varepsilon \iff |f(p)|<\varepsilon$, for any $\varepsilon>0$. This means that $f(p)=0$, which by definition means $p \in Z(f)$.

Thus, $Z(f)$ contains all of its limit points.

(II) Observe that: $f(Z(f))=\{0\}$.

($\subseteq$) Let $y \in f(Z(f))$. By definition, $Z(f)=\{p \in X : f(p)=0\}$.

So $y \in f(Z(f)) \Rightarrow \exists z \in Z(f)$ s.t $f(z)=y$. But $z \in Z(f)$ means $f(z)=0$. Therefore $y=0$, so $y \in \{0\}$.

($\supseteq$) Clearly $0 \in f(Z(f))$, because $z \in Z(f)$ means $f(z)=0$

($\subseteq$) & ($\supseteq$) $\Rightarrow f(Z(f))=\{0\}$.

We know that $\{0\}$ is closed (it has no limit points).

By theorem 4.8 (and its corollary), since f is continuous and $\{0\}$ is closed we must have that $f^{-1}(\{0\})$ ~~is~~ closed.

But by previous observation, $f^{-1}(\{0\}) = f^{-1}(f(Z(f))) = Z(f)$, so that $Z(f)$ must be closed.

1) Let f and g be continuous mappings of a metric space X into a metric space Y and let E be a dense subset of X.

Prove that: (I) $f(E)$ is dense in $f(X)$.

(II) If $g(p)=f(p)$ $\forall p\in E$, Then $g(p)=f(p)$ $\forall p\in X$.

Pf: (I) We want to show that if $x\in f(E)$, then x is a limit point of $f(X)$ or $x\in f(X)$ or both.

quivalently, let us show that if $y\in f(X)$ and $y\notin f(E)$ then y is a limit point of $f(E)$. So, suppose: $y\in f(X)$ and $y\notin f(E)$. then, by definition of $f(X)$, here exists $x\in X$ s.t. $f(x)=y$. claim: $x\notin E$. Pf: Suppose $x\in E$. hen $f(x)\in f(E)$, but $f(x)=y\notin f(E)$, a contradiction □. Hence $x\notin E$.

ut E is dense in X so $x\notin E \Rightarrow x$ is a limit point of E.

herefore, there exists a sequence in E, $\{x_n\}\subset E$ such that $\lim_{n\to\infty} x_n = x$. Now, use the hypothesis that f is continuous, $\lim_{n\to\infty} f(x_n)=f(x)=y$. Note that $y\notin f(E)$ and that for any ighbourhood of arbitrary radius $r>0$ $N_r(y)\setminus\{y\}\cap f(E)\neq\Phi$ because $f(x_n)\in N_r(y)\setminus\{y\}$ and $f(x_n)\in f(E)$, for n large enough. thus, is a limit point of $f(E)$.

) the idea here is to use the fact already proven and construct a sequence $\{x_n\}$ such that $x_n\to p$ as $n\to\infty$. We can do this ecause E is dense in X. But f is continuous, therefore: $f(x_n)\to f(p)$ and $g(x_n)\to g(p)$. By hypothesis $f(p)=g(p)$.

erefore $\lim_{n\to\infty} f(x_n)=f(p)=g(p)=\lim_{n\to\infty} g(x_n)$.

t $p\in X$ was arbitrary, so $f(p)=g(p)$, $\forall p\in X$.

(6) Suppose E is compact. Prove that f is continuous on E if and only if its graph is compact.

Pf: By definition $graph(f) = \{(x, f(x)) \mid x \in E\} = G(f)$.

($\Rightarrow$) Suppose E is compact and f is continuous on E.

We want to prove that $G(f)$ is compact. Take an open cover of $G(f)$, $\{G_\alpha\}$, for some set of indices α. Then $G(f) \subset \bigcup_\alpha G_\alpha$. But by hypothesis E is compact. Hence, Given an open cover of E, $\{A_\beta\}$, for some set of indices we can always extract a finite subcover s.t. $E \subset \bigcup_{i=1}^{n} A_{\beta i}$. Moreover, by theorem 4.14, since f is continuous. and E is compact we can conclude that $f(E)$ is compact. Again, we have a finite subcover for any open cover of $f(E)$, say $\{B_\gamma\}$ is an open cover.

Therefore, each piece of $G(f)$ can be finitely covered given an open cover for each piece. Combine these to get the result we want, i.e., let $\{A_\beta \times B_\alpha\}$ be an open cover of $G(f)$, that is then $\bigcup_{i=1}^{n} A_{\beta_i} \times B_{\alpha_i}$ is a finite subcover of $G(f)$,

$$\text{If } G(f) \subset \bigcup_{\beta, \alpha} A_\beta \times B_\alpha \Rightarrow G(f) \subset \bigcup_{\substack{i=1 \\ j=1}}^{\substack{n \\ m}} A_{\beta_i} \times B_{\alpha_j}$$

($\Leftarrow$) Suppose E is compact and $G(f)$ is compact.

We want to prove that f is continuous on E. Take x and $x_n \in E$ such that $\lim_{n\to\infty} x_n = x$. E compact $\Rightarrow E$ is closed $\Rightarrow E$ contains all of its limits points $\Rightarrow x \in E$.

The goal now is to show that $\lim_{n\to\infty} f(x_n) = f(x)$

Suppose that $\lim_{n\to\infty} f(x_n) \neq f(x)$. then $\exists\, \varepsilon > 0 : \forall\, \delta > 0 : \forall e \in$ $0 < d_X(x_n, x) < \delta$ and $d_Y(f(x_n), x) > \varepsilon$. But if this is the

CASE, then we could construct an open cover of $G(f)$ that contains no open subcover, a contradiction. therefore, $\lim_{n\to\infty} f(x_n) = f(x)$, for any arbitrary sequence in E, which means that f is continuous on E.

) If $E \subset X$ and if f is a function defined on X, the restriction of f TO E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$.
Define f and g on $\mathbb{R}^2$ by: $f(0,0) = g(0,0) = 0$.
$f(x,y) = xy^2/(x^2+y^4)$ and $g(x,y) = xy^2/(x^2+y^6)$ if $(x,y) \neq (0,0)$.
Prove that: (I) f is bounded on $\mathbb{R}^2$, (II) g is unbounded in every neighborhood of $(0,0)$ and (III) f is not continuous at $(0,0)$ (VI) the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous.

Pf: (I) $(x-y^2)^2 \geq 0$ (a square number is always positive)
$(x-y^2)^2 = x^2 - 2xy^2 + y^4 \geq 0 \Rightarrow x^2+y^4 \geq 2xy^2 \Rightarrow \frac{2xy^2}{x^2+y^4} \leq 1$, provided that $x \neq 0$ and $y \neq 0$. Hence, $\frac{xy^2}{x^2+y^4} \leq \frac{1}{2}$, and if $x=y=0$, then $f(x,y) = \frac{xy^2}{x^2+y^4} = 0 \leq \frac{1}{2}$. therefore, f is bounded by $\frac{1}{2}$, on $\mathbb{R}^2$.

) g is unbounded because
$g(\frac{1}{n^3}, \frac{1}{n}) = \frac{\frac{1}{n^3 n^2}}{\frac{1}{n^6}+\frac{1}{n^6}} = \frac{\frac{1}{n^5}}{\frac{2}{n^6}} = \frac{n^6}{2n^5} = \frac{n}{2}$ which is unbounded in every neighborhood of $(0,0)$. the sequences $\frac{1}{n^3}$ and $\frac{1}{n}$ can get as close as we want to zero.

) f is not continuous at $(0,0)$ because if we approach the limit by the parabola $y^2 = x$ we get $\lim_{y\to 0} g(y^2, y) =$
$\lim_{y\to 0} \frac{y^2 y^2}{y^4+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2} \neq 0 = g(0,0)$.

) $\lim_{x\to 0} f(x,mx) = \frac{xm^2x^2}{x^2+m^4x^4} = \frac{\cancel{x^2}(m^2x)}{\cancel{x^2}(1+m^4x^2)} = \frac{m^2x}{1+m^4x^2} \to 0$ as $x \to 0$ by L'Hopital

likewise $\lim_{x\to 0} g(x,mx) = \frac{xm^2x^2}{x^2+m^6x^6} = \frac{\cancel{x^2}(m^2x)}{\cancel{x^2}(1+m^6x^6)} = \frac{m^2x}{(1+m^6x^6)} \to 0$ as $x \to 0$ by L'Hopital

lines that do not go through origin are already continuous by composition of polynomials